Condensed Matter Journal Club

Interface dynamics prospect of directed avalanche systems

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The dynamics of a directed avalanche systems can be mapped to the dynamics of an interface model in one lower dimension. This allows us to understand the scaling behavior of avalanche systems from the prospective of their underlying interface dynamics. In this talk we'll introduce this mapping and discuss its application to a sandbox model.

1. criticality in nature
2. sandpile models
3. sandbox model
4. numerical results
5. mapping to interface model
6. correction to KPZ scaling
7. irrelevant operator
8. surface scar and rounding

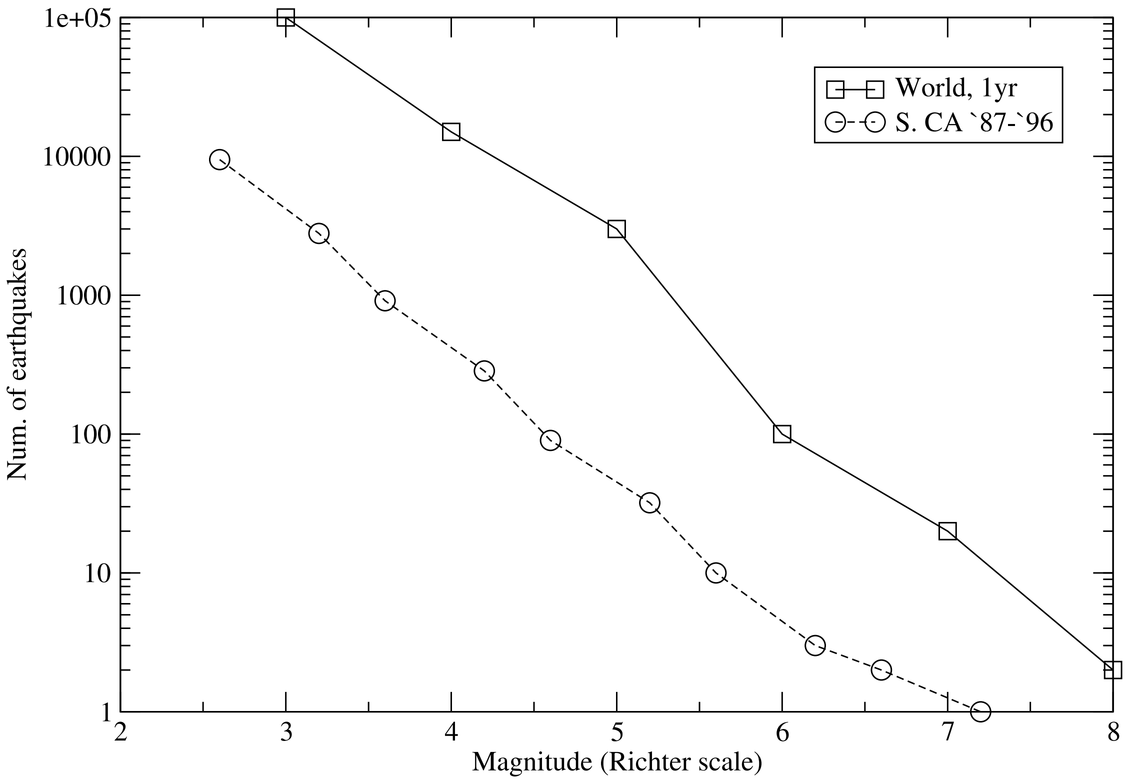
- earthquake magnitudes
- monthly variations of cotton prices
- extinction rates for four-million-year periods
- coast of Norway
- population of cities
- frequencies of words

Per Bak, *How nature works*, Copernicus, New York (1996)

Gutenberg-Richter's rule

B. Gutenberg and C. F. Richter, Seismicity of the earth, Special papers (Geological Society of America); 34 (1941)

$$\log N(M) = a - bM$$



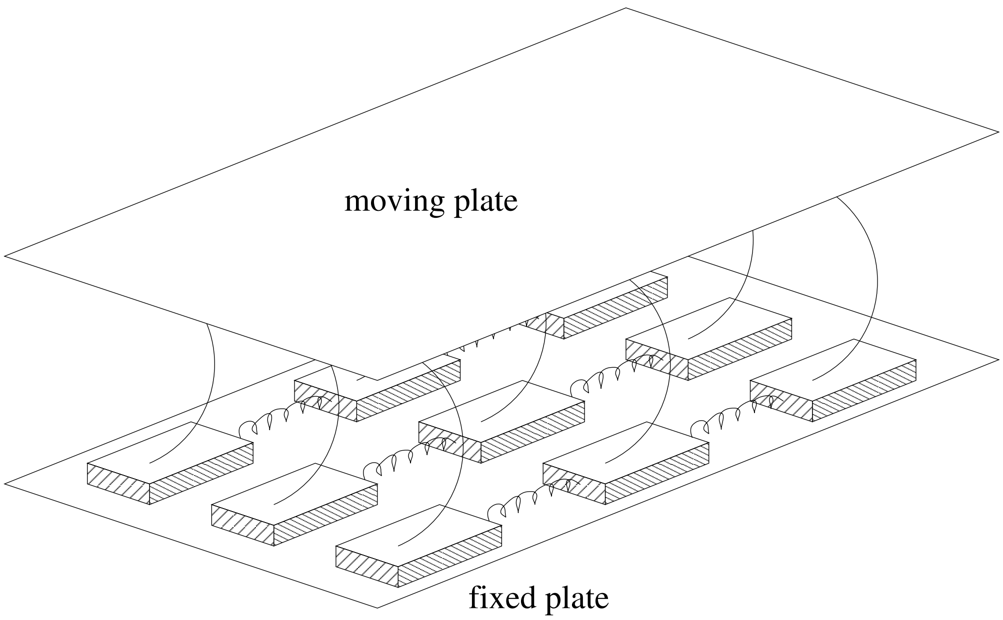
$b \approx 1$

spring-block model of earthquake fault

R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. 57, 341 (1967)

J. M. Carlson and J. S. Langer, Properties of earthquakes generated by fault dynamics, Phys. Rev. Lett. 62, 2632 (1989)

J. M. Carlson and J. S. Langer, Mechanical model of an earthquake fault, Phys. Rev. A 40, 6470 (1989)



Sandpile models

P. Bak, C. Tang, and K. Wiesenfeld, *Self-organized criticality*, Phys. Rev. A 38, 364 (1988)

stability condition

$$h_i < h_c$$

toppling rule for unstable site i

$$h_i \to h_i - \sum_{<i,j>} 1$$

$$h_j \to h_j + 1$$

for all neighboring pairs $< i, j >$.

driving rule: randomly choose k,

$$h_k \to h_k + 1$$

Illustration of one avalanche

1	2	0	2	3
2	3	2	3	0
1	2	3	3	2
3	1	3	2	1
0	2	2	1	2

1	2	0	2	3
2	3	2	3	0
1	2	4	3	2
3	1	3	2	1
0	2	2	1	2

1	2	0	2	3
2	3	3	3	0
1	3	0	4	2
3	1	4	2	1
0	2	2	1	2

1	2	0	2	3
2	3	3	4	0
1	3	2	0	3
3	2	0	4	1
0	2	3	1	2

1	2	0	3	3
2	3	4	0	1
1	3	2	2	3
3	2	1	0	2
0	2	3	2	2

1	2	1	3	3
2	4	0	1	1
1	3	3	2	3
3	2	1	0	2
0	2	3	2	2

1	3	1	3	3
3	0	1	1	1
1	4	3	2	3
3	2	1	0	2
0	2	3	2	2

1	3	1	3	3
3	1	1	1	1
2	0	4	2	3
3	3	1	0	2
0	2	3	2	2

1	3	1	3	3
3	1	2	1	1
2	1	0	3	3
3	3	2	0	2
0	2	3	2	2

Avalanche distributions of sandpile models

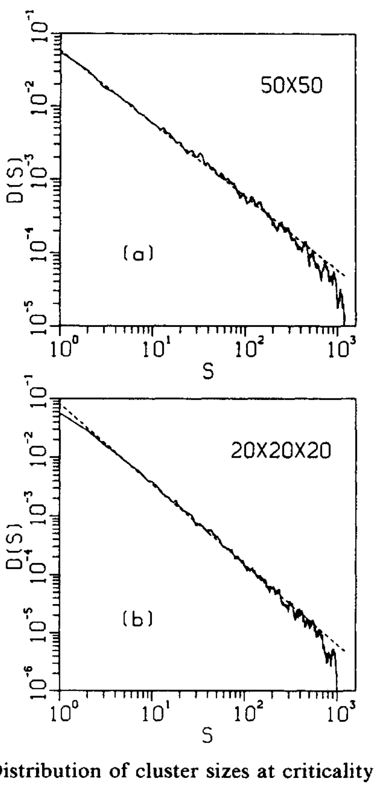


FIG. 3. Distribution of cluster sizes at criticality in two and three dimensions computed as described in the text. The data have been coarse grained. (a) 50×50 array, averaged over 200 samples. The dashed line is a straight line with slope -1.0; (b) $20\times 20\times 20$ array, averaged over 200 samples. The dashed straight line has a slope -1.37.

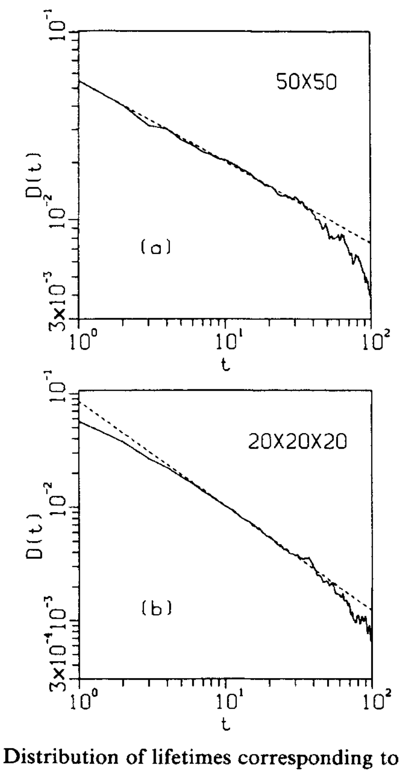
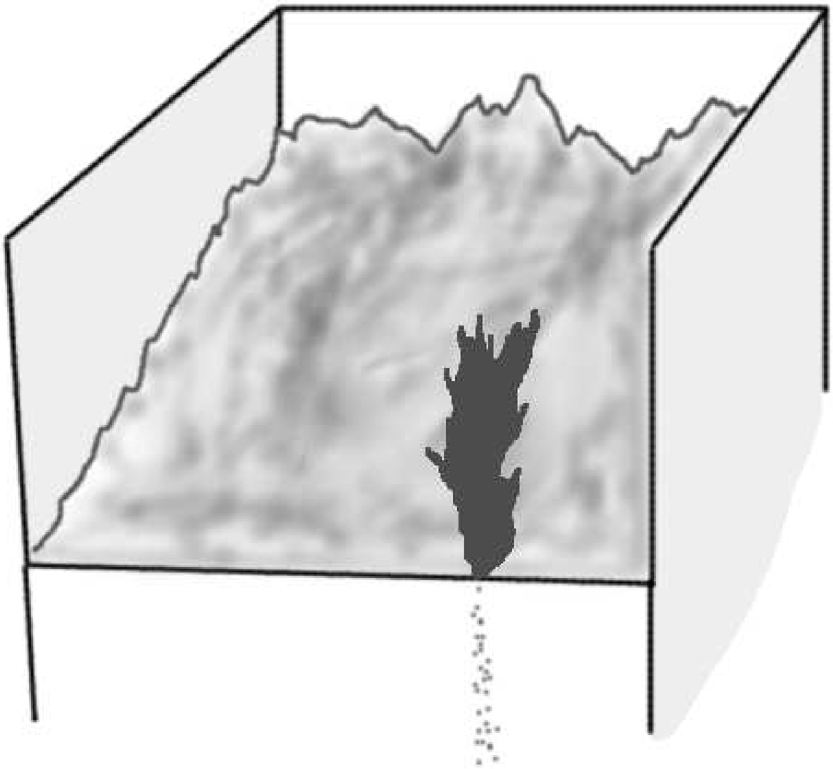


FIG. 4. Distribution of lifetimes corresponding to Fig. 3. (a) For the 50×50 array, the exponent $\alpha \approx 0.43$ yields a $1/f$ noise spectrum $f^{-1.57}$; (b) $20\times 20\times 20$ array, $\alpha \approx 0.92$, yielding an $f^{-1.08}$ spectrum.

Sandbox picture



Sandbox model

stability condition

$h(x, y) \leq \min[h(x-1, y-1), h(x+1, y-1)] + 1$

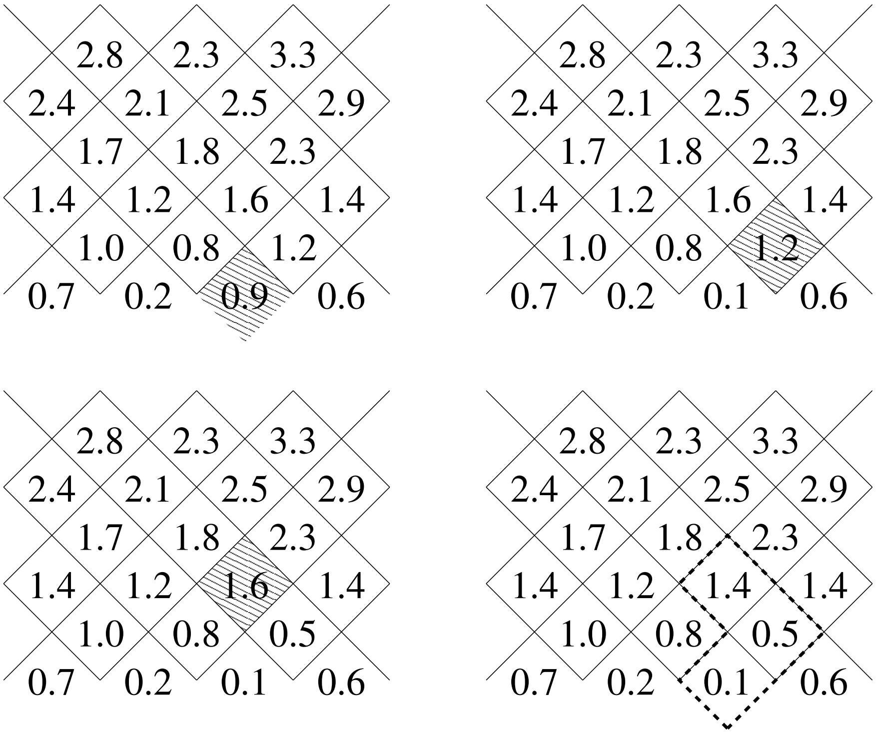
toppling rule

$h(x, y) \to \min[h(x-1, y-1), h(x+1, y-1)] + \eta_i(x, y)$

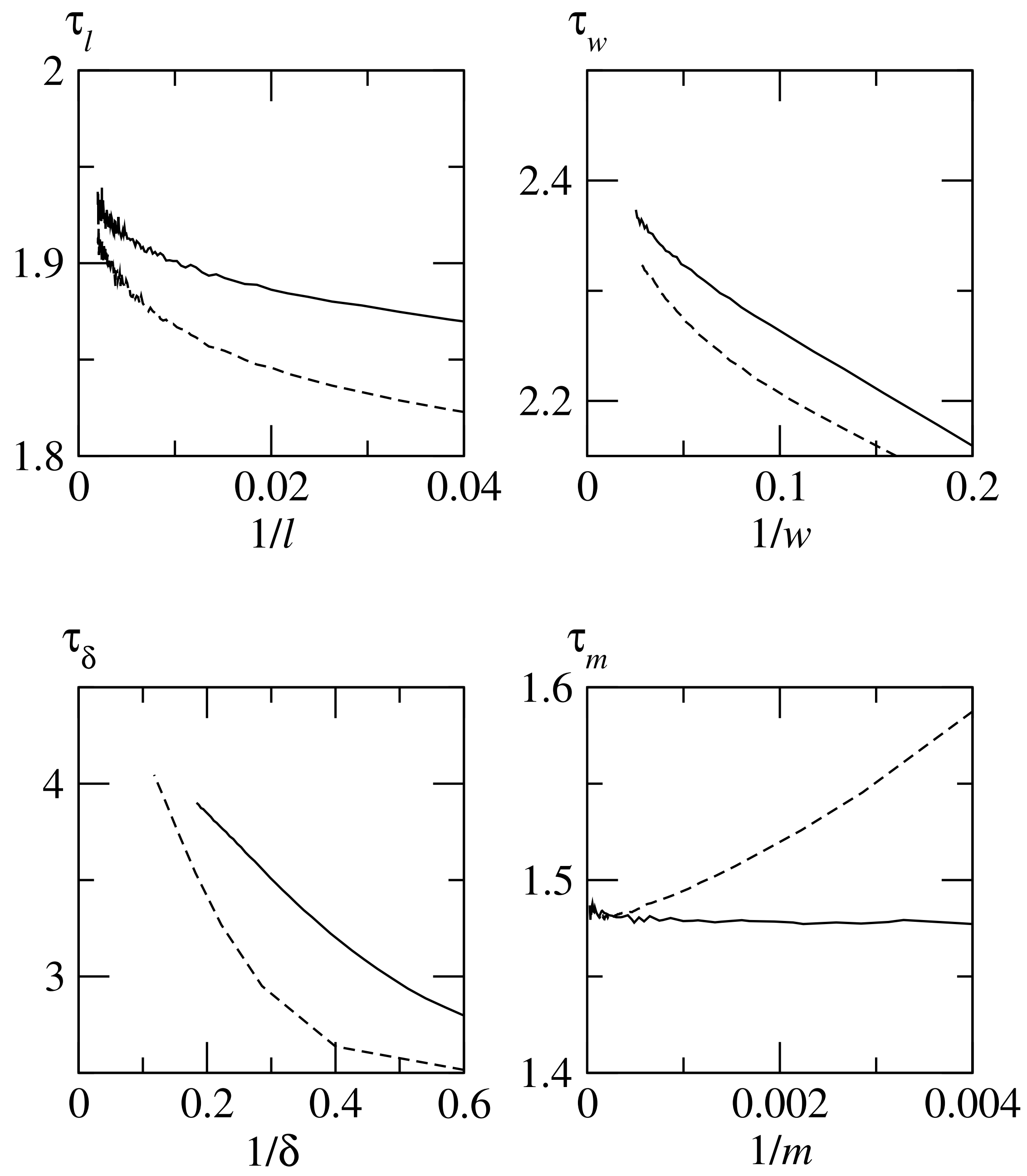
driving rule: find $h(x_i, 0) = \max_x h(x, 0)$,

$$h(x_i, 0) \to h(x_i, 0) - \eta_i$$

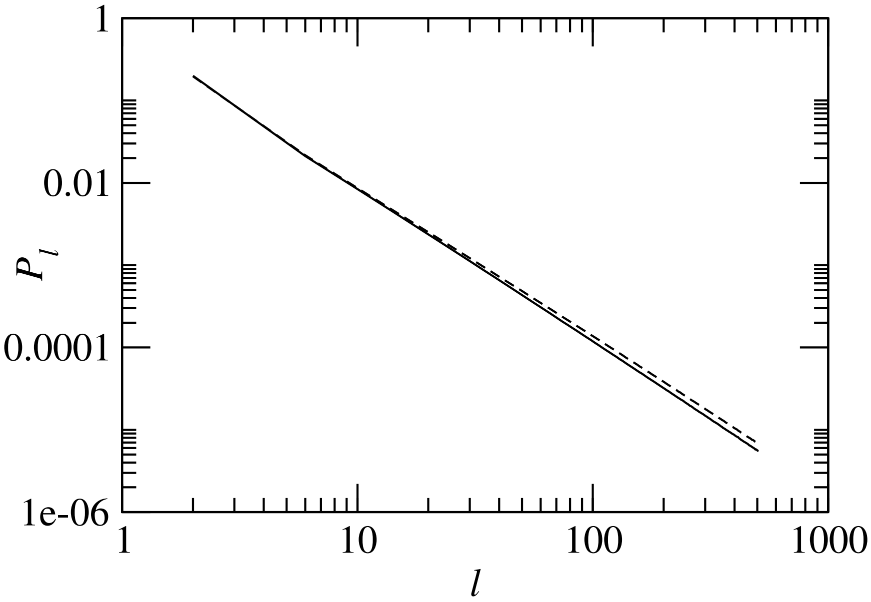
($0 \leq \eta \leq 1$, uniform, uncorrelated)

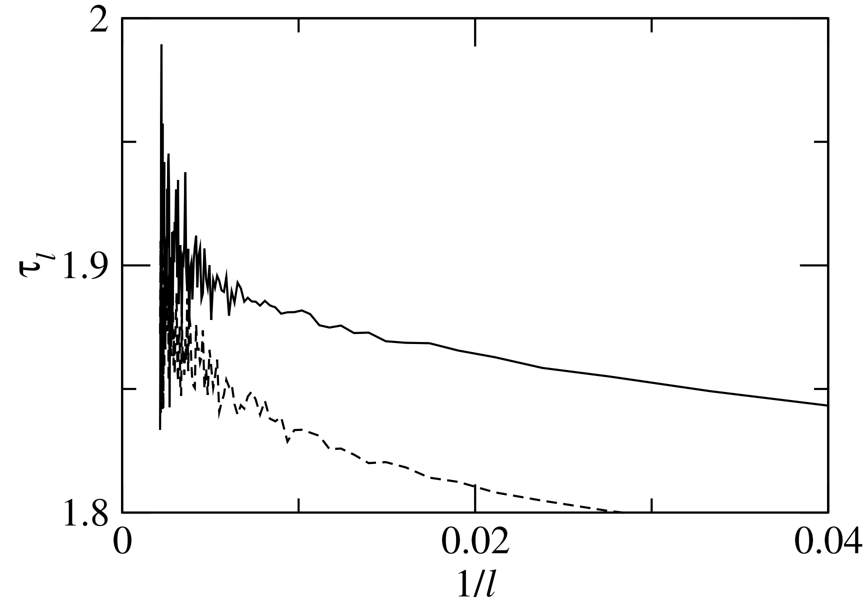


Avalanche distributions

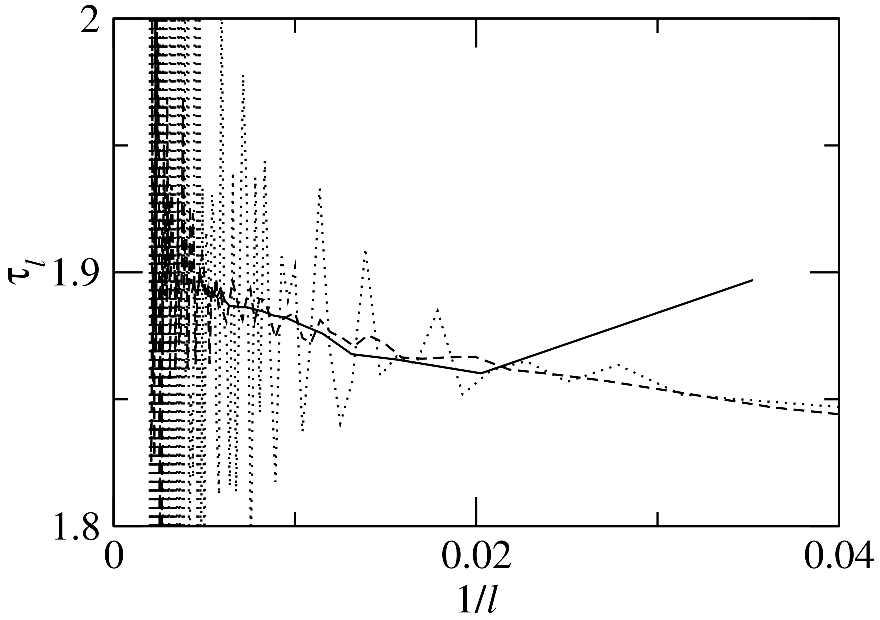


Finite size scaling





point vs interval fit

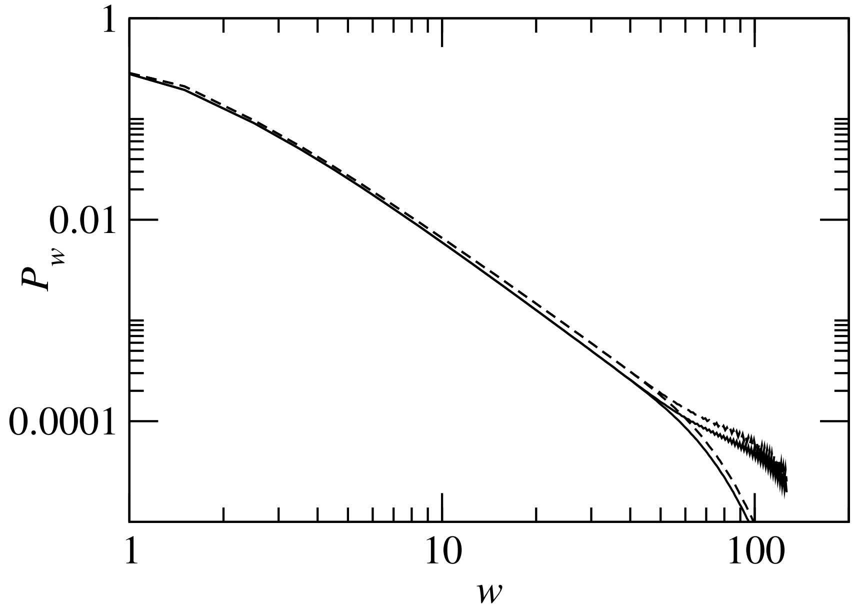


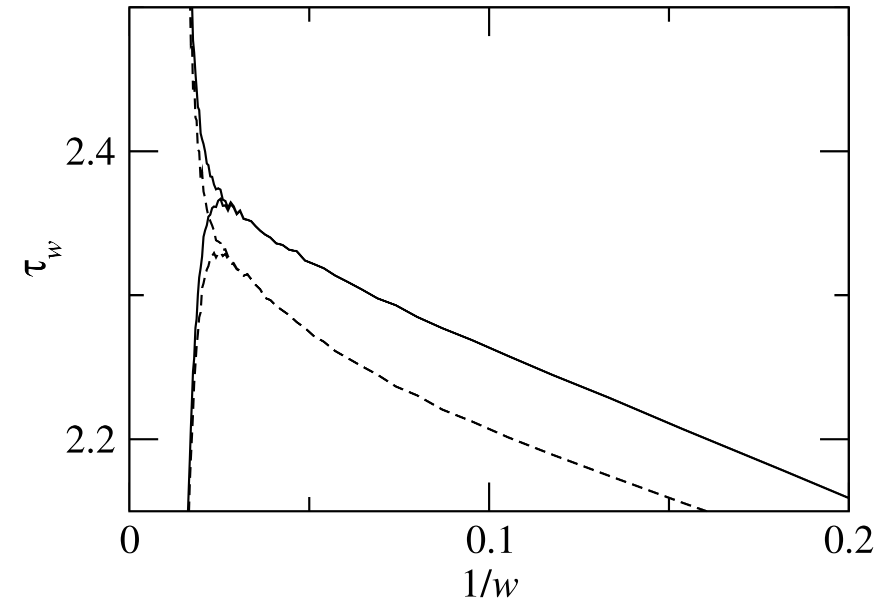
$$P_l(l) = A l^{-\tau_l}$$

$$Q_l(l) = \int_l^\infty P_l(l') dl' = \frac{A}{\tau_l - 1} l^{-\tau_l + 1}$$

$$\Rightarrow \tau_l(l) = 1 + \frac{P_l(l)}{Q_l(l)}$$

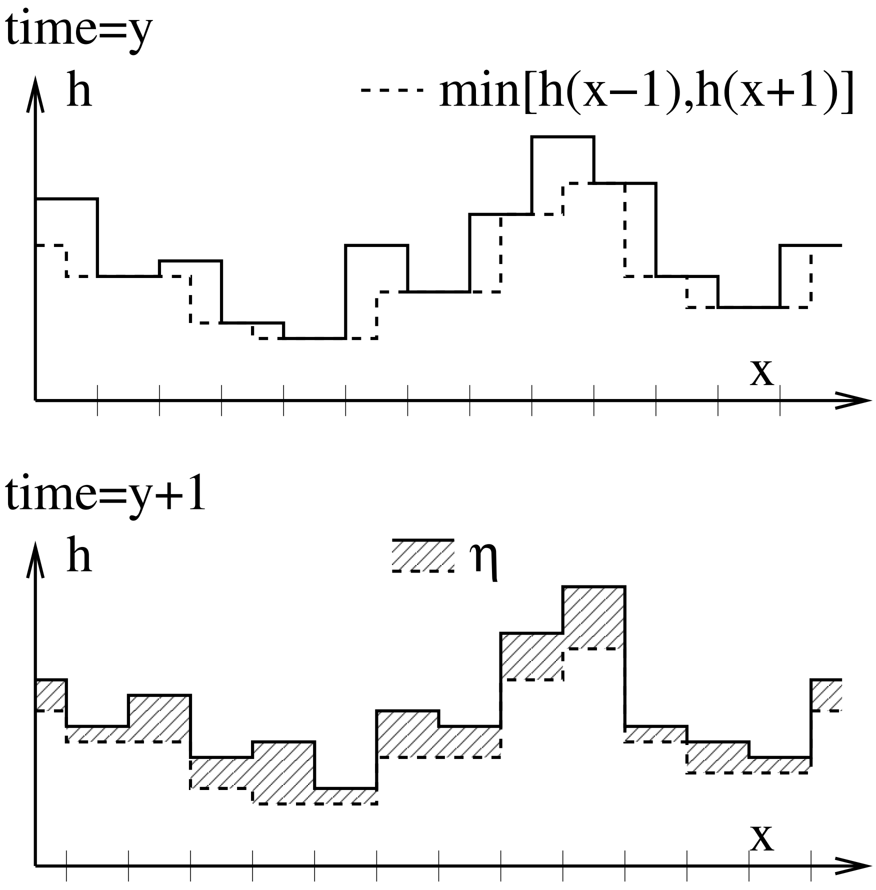
Boundary effect





$$
\begin{aligned}
h(x, y+1) &= \frac{1}{2}\left[h(x+1, y)+h(x-1, y)\right] \\
&- \frac{1}{2}\left|h(x+1, y)-h(x-1, y)\right|+\eta(x, y)
\end{aligned}
$$

$$
\Rightarrow \frac{\partial h}{\partial y}=\nabla^{2} h-\frac{\lambda}{2}(\nabla h)^{2}+\eta
$$



exponents relations

$$P(l,w,\delta) = b^{-\sigma} P(b^{-z} l, b^{-1} w, b^{-\alpha}\delta)$$

$$\tau_l = \frac{\sigma - 1 - \alpha}{z},\ \tau_w = \sigma - z - \alpha,\ \tau_\delta = \frac{\sigma - 1 - z}{\alpha}$$

$$z = \frac{\tau_w - 1}{\tau_l - 1},\ \alpha = \frac{\tau_w - 1}{\tau_{\delta - 1}},\ \sigma = \tau_w + z + \alpha$$

for mass, $m \sim lw\delta$

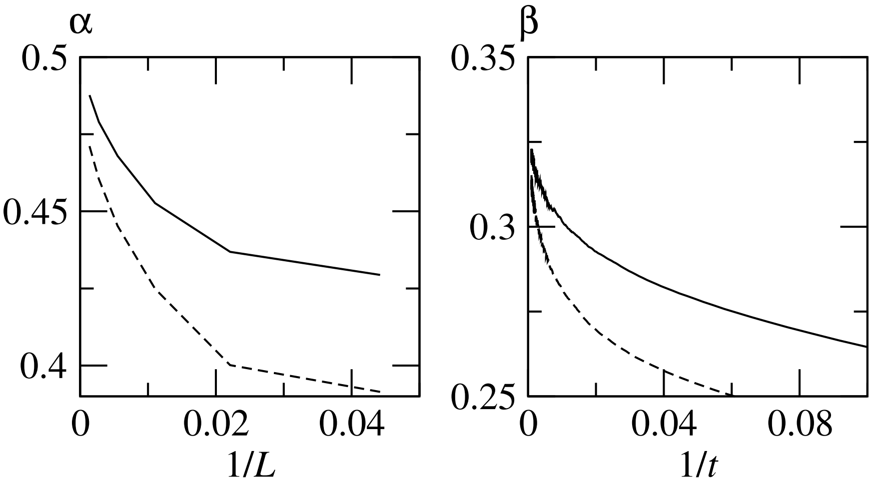
$$\tau_m = \frac{\sigma}{1 + z + \alpha}$$

conservation:

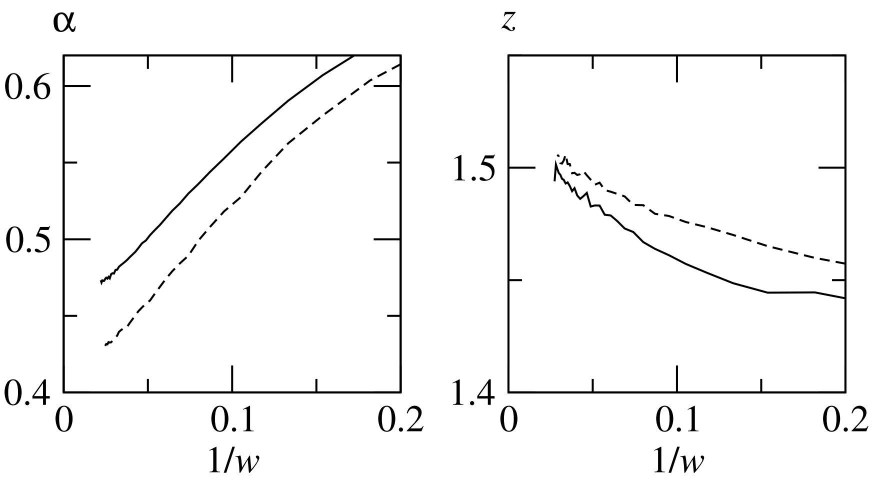
$$\langle m \rangle = \frac{1}{2} L_y$$

$$\Rightarrow \sigma = 2 + z + 2\alpha$$

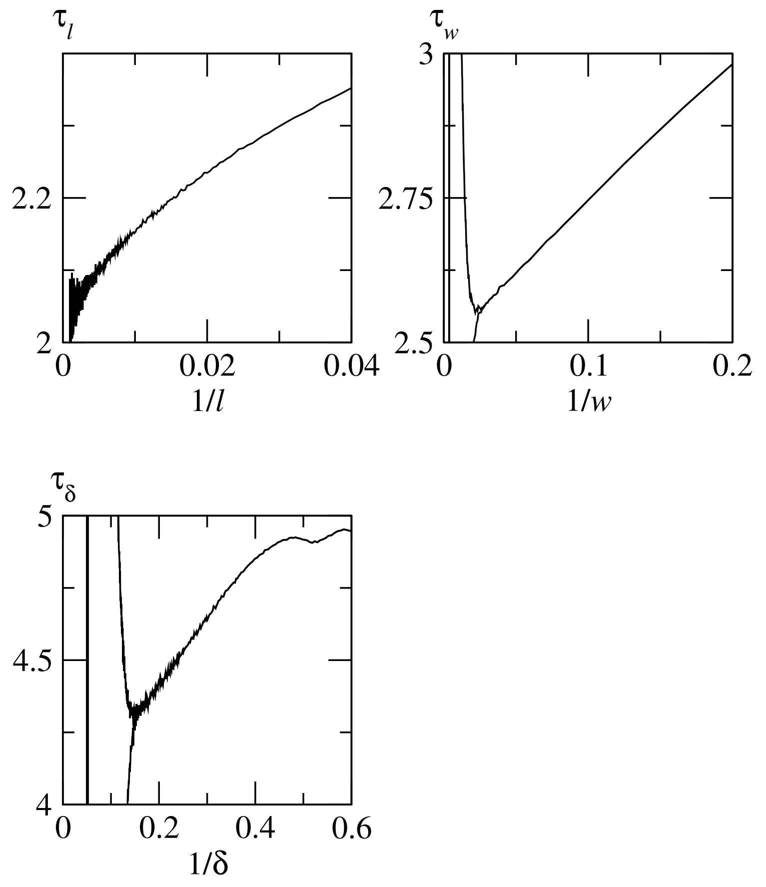
interface exponents

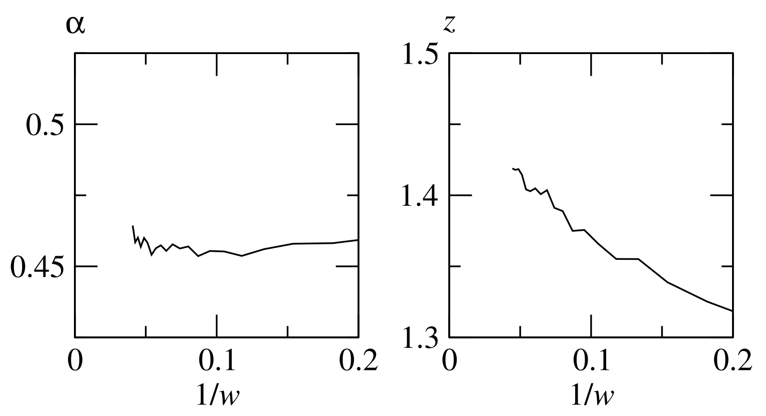


stationary sand surface



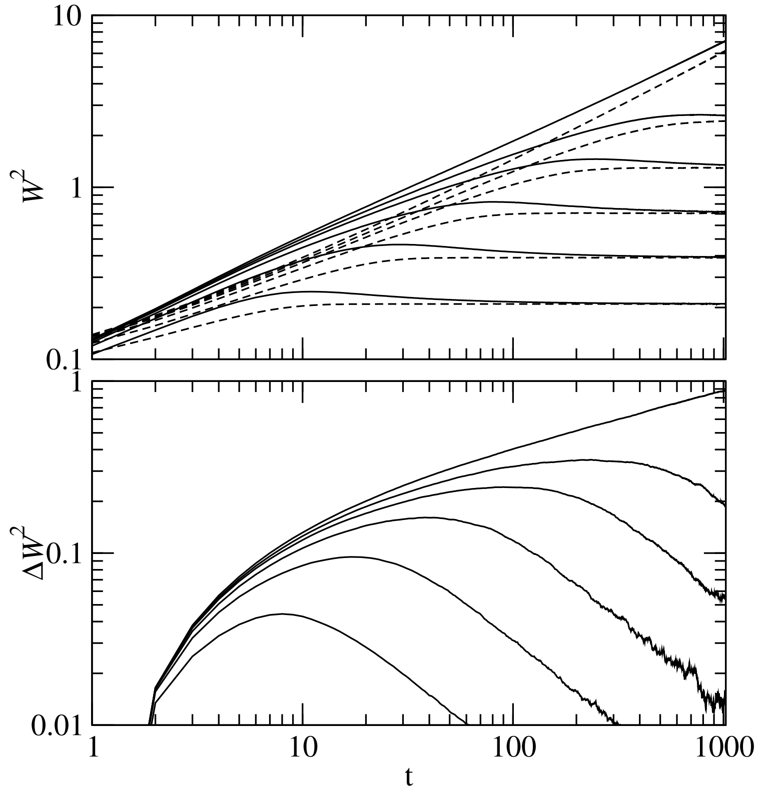
avalanche on fresh surface

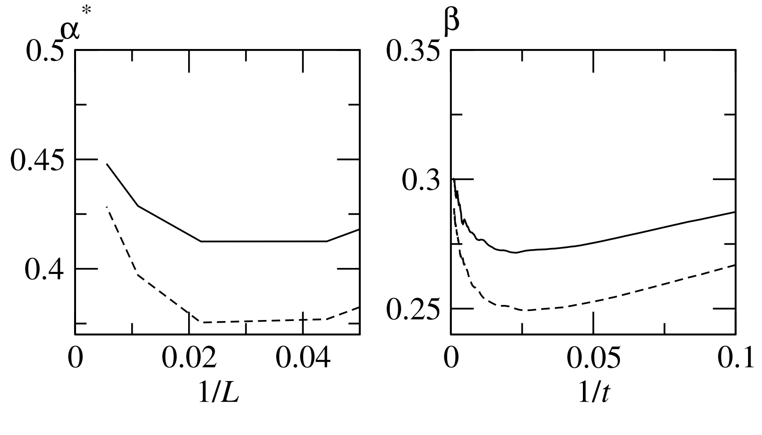




stationary sand surface width

from bottom up, L_x is 8, 16, 32, 64, 128, ∞





Correction to scaling

with irrelevant operator O_{sc},

$$\frac{\partial h}{\partial t} = \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + uO_{sc}$$

scaling ansatz:

$$W^2(L_x, t, u) = b^{2\alpha} W(b^{-1}L_x, b^{-z}t, b^{y_{ir}}u)$$

taking limit $L_x \to \infty$ and asumming $y_{ir} < 0$

$$\begin{aligned} W^2(t, u) &= t^{\frac{2\alpha}{z}} S(t^{\frac{y_{ir}}{z}} u) \\ &= t^{\frac{2\alpha}{z}} \left[S(0) + t^{\frac{y_{ir}}{z}} S'(0) + \cdots \right] \end{aligned}$$

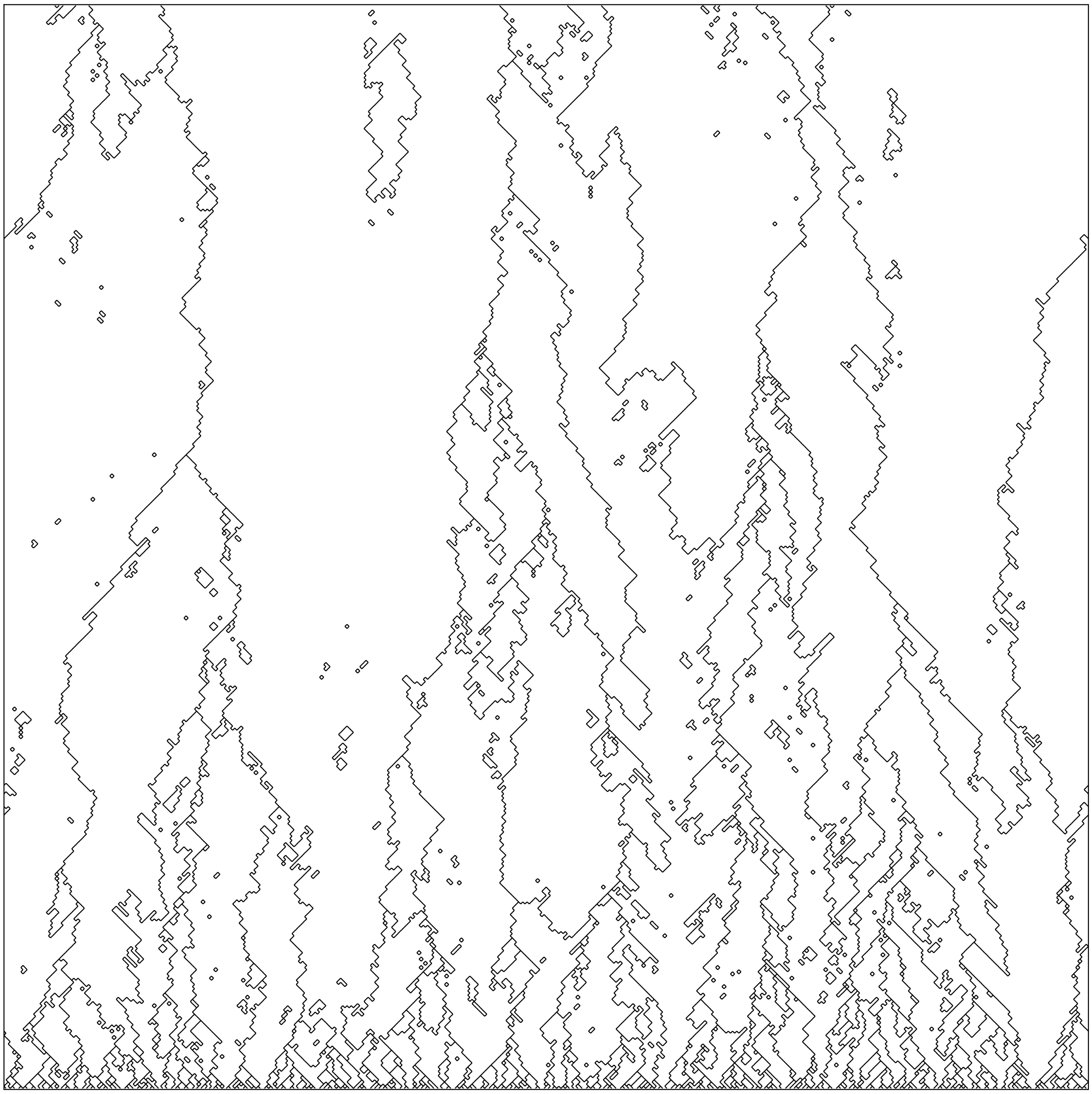
since $\delta W^2 \sim t^{1/3} \sim \sqrt{W^2}$, $y_{ir} = -\alpha$

$$\Rightarrow O_{sc} \sim b^{-x_{sc}} = b^{-z}$$

g: age field on the surface

$$O_{sc} \sim \frac{\widehat{e}_y \cdot \nabla g}{|\nabla g|}$$

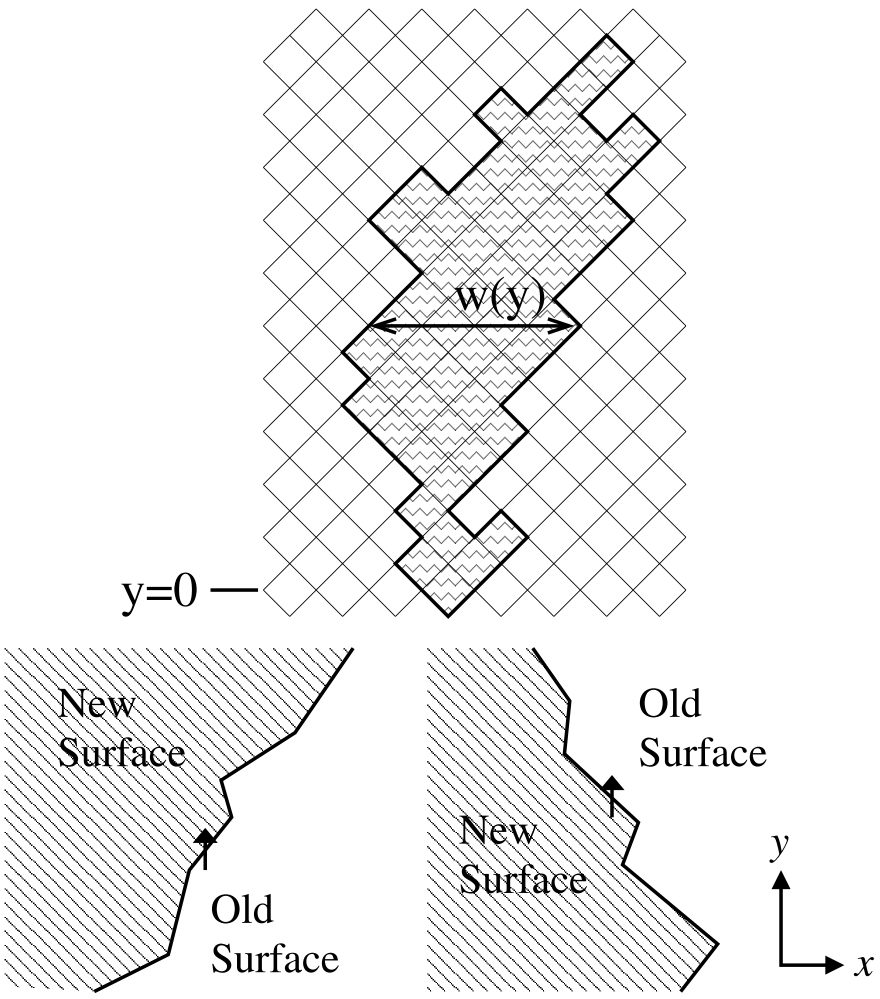
scars on stationary sand surface



Equation of motion for scars

$s_t(y) \equiv \langle O_{sc} \rangle$, t: avalanche time

$$L_x \frac{\partial s_t(y)}{\partial t} = \left\{ \Delta \left[\theta_t^L(y) \right] - \Delta \left[\theta_t^R(y) \right] \right\} - w_t(y) s_t(y)$$



Stationary solution $s(y) = \overline{s_t(y)}$

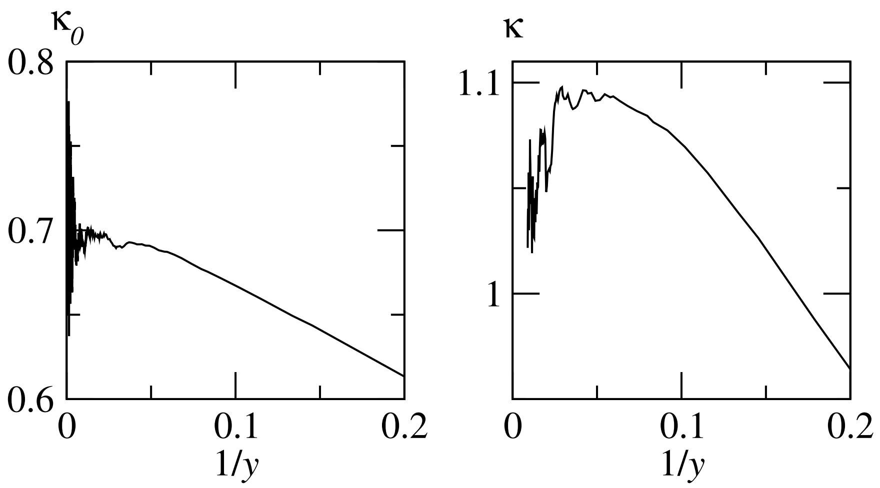
$$\begin{aligned}
\overline{w_t(y)s_t(y)} &= \overline{\Delta\left[\theta_t^L(y)\right] - \Delta\left[\theta_t^R(y)\right]} \\
&\approx a\overline{\left[\theta_t^L(y) - \theta_t^R(y)\right]} \\
\overline{w_t(y)}s(y) &\approx a\frac{\partial \overline{w_t(y)}}{\partial y}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow s(y) &\approx a\frac{\partial}{\partial y}\log\left[\overline{w_t(y)}\right] \\
&\sim y^{-1}
\end{aligned}$$

as $\overline{w_t(y)} \sim \int_y^\infty w(y)P(l)dl \sim y^{1/z-\tau_l+1} \sim y^{-1/3}$

Avalanche rounding exponent κ

$$\begin{aligned} \overline{\frac{\partial h(y)}{\partial y}} &= s_{\mathsf{f}}(y) + s(y) \\ &\approx v_0 + c_1 y^{-\kappa_{\mathsf{f}}} + c_2 y^{-\kappa} \end{aligned}$$



$\kappa_{\mathsf{f}} = 2/3$ and $\kappa = 1$

Conclusions

- directed system with local dynamics can be mapped to interface dynamics in one lower dimension

- iterated avalanche process results in correlated sampling in the ensemble space and modifies the scaling behavior

- compact avalanche clusters localized modification to the edges of the avalanches (scars)

- this effect dies out with a universal power -1 of the distance from the driving boundary and results in a strong correction but not fundamental change to the scaling behaviors